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Finite density effects in Hosotani mechanism and a vacuum gauge ball

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Abstract

We consider the finite density effects of the fermion with $U(1)$ gauge symmetry in Hosotani mechanism. We construct a vacuum gauge ball, a new kind of non-topological soliton, and investigate their properties numerically. We find the relations between the physical quantities.

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I. INTRODUCTION

Recently, there has been much interest in symmetry breaking mechanisms in theories with extra dimensions. (For a recent review, see Ref. [1].) In particular, Hosotani mechanism [2, 3] attracts much attention and has been studied by many authors. In this mechanism, the vacuum expectation value (VEV) of the gauge field configuration on the multiply-connected space, which is related with the Wilson line element on the space, plays an important role. The Wilson line elements on the space become dynamical degrees of freedom and parametrize degenerate classical vacua. The quantum effects (in terms of the one-loop effective potential) lift the degeneracy and determine the physical vacua. If the effective potential has a minimum corresponding to a non-trivial configuration, the symmetry is spontaneously broken or enhanced.

On the other hands, in cosmological context, the analysis of high temperature or high density effects is essential to understand the phase transition during the cosmological evolution. In ordinary field theories including spontaneous symmetry breaking, the symmetry is expected to be restored at high temperature [4] or high density [5]. In Hosotani mechanism, the finite density and high temperature effects have been investigated in [6, 7, 8, 9, 10]. One of the authors found that the symmetry can be broken or restored by high density effects of the degenerate fermion [6]. The various "phases" may exist inside an object of high density.

In this paper, we consider the finite density effects for the degenerate fermion in Hosotani mechanism. Instead of studying breaking and restoration of symmetry by the effects, we construct an object by similar methods to Fermi ball (F-ball) [11, 12], and Q-ball [13] such as a kind of non-topological soliton (NTS) [14], which may be related to the important clue to solve problems in cosmology, for example, dark matter problem. We call the new object "vacuum gauge ball".

NTS is formed as a stable solution whose boundary condition at infinity is expressed as the vacuum state. Q-ball, which is a kind of NTS, is stabilized by conservation of global $U(1)$ charge. One of the characters of the scalar potential associated with them is that there is a barrier between the symmetric and broken phases. The conceptual construction of F-ball, on the other hands, consists of three steps. The first step is that there are two almost degenerate vacua (one is true vacuum and another is false one), which appear in the spontaneous breaking of an approximate Z_2 symmetry, which is called "biased Z_2 symmetry"

in [11]. At the phase transition, a domain wall is produced between two vacua and zero-mode fermions are captured in the domain wall. The next is that if the true vacuum is energetically favored, then a region of false vacuum gets diminished and continues to shrink due to the surface energy as well as the volume energy inside the domain wall. Finally, the Fermi energy stops the shrinkage and keeps the dynamical balance. Such a bag of the false vacuum with zero-mode fermions caught in the surrounding wall is called F-ball.

If Hosotani mechanism is applied to the case of finite density, it is also possible to realize the similar situations to the case of F-ball and Q-ball: the thermodynamic potential (or the effective potential) in Hosotani mechanism has a barrier between their vacua, appeared in case of Q-ball. Further, the two vacua are almost degenerate with the energy density difference, parametrized by (chemical potential) $\times$ (circumference of the extra dimension). When the vacuum gauge field forms a domain wall configuration, the vacuum gauge ball is constructed, owing to the dynamical balance between the shrinking force due to the energy difference of the two vacua and the expanding force due to the Fermi energy, appeared in such case as F-ball.

In this setting, we investigate the relationship between the mass, the radius and the particle number of the vacuum gauge ball in $M_3 \times S^1$ and $M_4 \times S^1$ space-times.

The organization of this paper is as follows. In the next section, in order to consider the finite density effects in Hosotani mechanism, we evaluate the thermodynamic potential for the vacuum gauge field, using the zeta function regularization technique. We construct a vacuum gauge ball and investigate their property numerically in $M_3 \times S^1$ and $M_4 \times S^1$ space-times in section III. Finally, we devote section IV to discussion.

II. FINITE DENSITY EFFECTS IN HOSOTANI MECHANISM

In this section, we begin by considering a gauge theory with a Dirac fermion in $M_{(d+1)} \times S^1$ space-time. Let x^i ($i = 0, 1, \ldots, d$) and y label the $(d+1)$-dimensional Minkowski space-time and S^1 space whose circumference is L ($0 \leq y < L$), respectively, (μ, ν run over both i and y). In our model, we treat $U(1)$ gauge symmetry, for simplicity. Here we still call the mechanism (of determining the non-trivial Wilson element by quantum effects) as Hosotani mechanism, even if the symmetry is abelian (and is not broken). Generalizing to the cases of various gauge symmetries and topology of the extra dimensions are straightforward.

The Lagrangian is denoted by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}D_{\mu}\psi, \tag{1}$$

where the Maxwell field strength and the covariant derivative with the gauge coupling constant g are given by

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \tag{2}$$

$$D_{\mu} = \partial_{\mu} - igA_{\mu}, \tag{3}$$

respectively. Since S^1 is a non-simply connected space, one must specify the boundary conditions of the fields. The boundary conditions on the gauge field and the fermion field are chosen as follows

$$A_{\mu}(x, y+L) = A_{\mu}(x, y), \tag{4}$$

$$\psi(x, y+L) = e^{i\delta}\psi(x, y). \tag{5}$$

The phase $e^{i\delta}$ cancels from physical operators of a bilinear form $\bar{\psi}\psi$ but contributes to the boundary conditions.

In S^1 space, non-zero vacuum gauge field configuration is admitted. Then, we set the VEV of the gauge field configuration

$$\phi = \langle gLA_y \rangle, \tag{6}$$

where A_y denotes a component of the gauge field on S^1. If we just consider non-singular gauge transformation, non-zero $\langle gLA_y \rangle$ is inequivarent to the trivial vacuum gauge field $\langle gLA_y \rangle = 0$ unless $\langle gLA_y \rangle = 2\pi n$ (n: integer). Then one can only restrict $0 \le \phi < 2\pi$ by gauge transformations. Note that the VEV of the gauge field configuration ϕ plays the role of an "order parameter" and the quantum effects in terms of the effective potentials determine the order parameter. For instance, one finds the expression of the one-loop effective potential for the vacuum gauge field from the massless Dirac fermions

$$V_{\text{eff}} = -(\mathrm{tr}\mathbf{1})\frac{1}{2}\frac{N_f}{(LV_d)}\ln\det(D^2), \tag{7}$$

where V_d is the volume of d-dimensional space. N_f and $\mathrm{tr}\mathbf{1}(= 2^{[(d+2)/2]})$ are the number of a Dirac fermion and the number of components of the fermion, respectively. $[x]$ is the integral

part of x. If the effective potential for ϕ is minimized at a non-trivial configuration, then the fermions acquire dynamical masses from the quantum effects. The effective potential (7) can be evaluated by the zeta function regularization method [15, 17, 18],

$$\ln\det(D^2) = -\zeta'(0), \tag{8}$$

where $\zeta(s)$ is the zeta function, which is defined according to the field contents.

Now, we shall consider the finite density effects in Hosotani mechanism. For this purpose, we adopt the usual imaginary time formalism, according to the standard techniques in finite-temperature field theory [16]: the time coordinate is Wick rotated to the Euclidean time $\tau = it$ $(0 \leq \tau < \beta)$ with boundary conditions on fields (periodic for bosons and anti-periodic for fermions). The chemical potential μ of the fermion field is introduced as the zeroth component of the covariant derivative to be modified by

$$D_0 = \partial_0 \to \partial_0 - iA_0, \qquad \text{where} \quad A_0 = -i\mu. \tag{9}$$

Employing the zeta function regularization method, the effective potential, or the thermodynamic potential Ω in statistical mechanics is expressed as the following form at temperature β^{-1} and chemical potential μ:

$$\beta\Omega = (\mathrm{tr}\mathbf{1})\frac{1}{2}\frac{N_f}{LV_d}\zeta'(0), \tag{10}$$

with

$$\begin{aligned} \zeta(s) &= \frac{2V_d}{\Gamma(s)}\int_0^\infty dt t^{s-1}\int\frac{d^dk}{(2\pi)^d} \\ &\quad \times \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\exp\left[-t\left\{k^2 + \left(\frac{(2n+1)\pi}{\beta} + i\mu\right)^2 + \left(\frac{2\pi l + \phi}{L}\right)^2\right\}\right], \end{aligned} \tag{11}$$

where, for simplicity, we have restricted the Dirac fermions to the case of periodic boundary condition on S^1, *i.e.*, $\delta = 0$ in (5). The factor "two" of the right hand side in (11) is the contributions from the degeneracy of the fermion in the case that the compact manifold is S^1. In general, the degeneracy of the fermion is given by

$$d_l(N) = \frac{2\Gamma(N+l)}{l!\Gamma(N)}, \tag{12}$$

for a sphere S^N case [19].

In order to carry out the calculation, it is convenient to use the following identity in terms of theta function [24]

$$\sum_{n=-\infty}^{\infty} \exp\left[-t\left(\frac{(2n+1)\pi}{\beta}+i\mu\right)^2\right] = \frac{\beta}{(4\pi t)^{1/2}}\left[1+2\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\exp\left(-\frac{\beta^2n^2}{4t}\right)\right]. \tag{13}$$

Then one can divide $\zeta(s)$ into two parts

$$\zeta(s) = \zeta_0(s)+\zeta_\beta(s), \tag{14}$$

$$\zeta_0(s) = \frac{2\beta}{(4\pi)^{1/2}}\frac{V_d}{\Gamma(s)}\int_0^\infty dt t^{s-\frac{3}{2}}\int\frac{d^dk}{(2\pi)^d}\sum_{l=-\infty}^{\infty}\exp\left[-t\left\{k^2+\left(\frac{2\pi l+\phi}{L}\right)^2\right\}\right], \tag{15}$$

$$\zeta_\beta(s) = \frac{4\beta}{(4\pi)^{1/2}}\frac{V_d}{\Gamma(s)}\int_0^\infty dt t^{s-\frac{3}{2}}\int\frac{d^dk}{(2\pi)^d}\sum_{l=-\infty}^{\infty}\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu) \times\exp\left[-t\left\{k^2+\left(\frac{2\pi l+\phi}{L}\right)^2\right\}-\frac{\beta^2n^2}{4t}\right], \tag{16}$$

where $\zeta_0(s)$ part contributes to the thermodynamic potential as the temperature-independent vacuum energy and $\zeta_\beta(s)$ part is the temperature-dependent contribution to the vacuum energy. In the following subsections, we calculate each part of the zeta function $\zeta(s)$ in turn.

A. calculation of $\zeta_0(s)$ part

In calculation of $\zeta_0(s)$ part, one can also use the following identity in terms of theta function in (15)

$$\sum_{l=-\infty}^{\infty}\exp\left[-t\left(\frac{2\pi l+\phi}{L}\right)^2\right] = \frac{L}{(4\pi t)^{1/2}}\sum_{l=-\infty}^{\infty}\exp\left[-\frac{L^2}{4t}l^2\right]\cos(l\phi), \tag{17}$$

where we have only adopted the real part of the right hand side in the identity because the imaginary part does not contribute to the thermodynamic potential.

Then, the thermodynamic potential contributed from the temperature-independent part,

which is denoted by Ω_0, is reduced to

$$\Omega_0 \equiv \frac{(\mathrm{tr}\mathbf{1})}{2\beta}\frac{N_f}{V_d L}\zeta_0'(0), \tag{18}$$

$$= (\mathrm{tr}\mathbf{1})\frac{N_f}{(4\pi)^{(d+2)/2}}\int_0^\infty dt t^{-\frac{d+2}{2}-1}\sum_{l=-\infty}^{\infty}\exp\left[-\frac{L^2}{4t}l^2\right]\cos(l\phi), \tag{19}$$

$$= (\mathrm{tr}\mathbf{1})\frac{2N_f}{\pi^{\frac{d+2}{2}}}\Gamma\left(\frac{d+2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{(Ll)^{d+2}}\cos(l\phi), \tag{20}$$

where in the second line (19), we have performed the integration of the momentum k and, in the third line (20), we have replaced the summation on l in the Kaluza-Klein modes as follows

$$\frac{L^2 l^2}{4t} \to t, \tag{21}$$

which causes the divergence at $l = 0$ and have omitted the divergence, which is the contributions to the cosmological constant. We assume that the cosmological constant of the true vacuum one obtains is set to zero by some mechanisms.

In $M_3 \times S^1$ space-time, if the ϕ is restricted to the case of $0 \leq \phi \leq \pi$, then the thermodynamic potential can be obtained as an analytical form

$$\Omega_0 = \frac{8N_f}{\pi^2 L^4}\sum_{l=1}^{\infty}\frac{\cos(l\phi)-(-1)^l}{l^4}, \tag{22}$$

$$= \frac{N_f}{6}\frac{1}{\pi^2 L^4}\left\{2\pi^2(\phi-\pi)^2-(\phi-\pi)^4\right\}, \tag{23}$$

here we have used the useful identities

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^4} = \frac{1}{48}\left[2\pi^2(x-\pi)^2-(x-\pi)^4-\frac{7}{15}\pi^4\right], \tag{24}$$

and $\sum_{l=1}^{\infty}\frac{(-1)^l}{l^4} = -\frac{7}{8}\zeta(4)$.

B. calculation of $\zeta_\beta(s)$ part

Next, we calculate the thermodynamic potential of $\zeta_\beta(s)$ part as the temperature-dependent contribution to the vacuum energy, which is denoted by Ω_β,

$$\begin{aligned}\Omega_\beta &\equiv (\mathrm{tr}\mathbf{1})\frac{1}{2\beta}\frac{N_f}{V_d L}\zeta'_\beta(0), &(25)\\ &= (\mathrm{tr}\mathbf{1})\frac{2}{(4\pi)^{\frac{d+1}{2}}}\frac{N_f}{L}\int_0^\infty dt t^{-\frac{d+1}{2}-1}\sum_{l=-\infty}^{\infty}\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\\ &\quad\times\exp\left[-t\left(\frac{2\pi l+\phi}{L}\right)^2-\frac{\beta^2 n^2}{4t}\right], &(26)\end{aligned}$$

where we have performed the integration of the momentum k. Moreover we divide three parts for the summation on l ($l=0$, plus and minus parts of l) and use the following integral representation of the modified Bessel function [20]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \exp\left(-t-\frac{z^2}{4t}\right)t^{-\nu-1}dt. \tag{27}$$

Then the thermodynamic potential contributed from the temperature-dependent part is reduced to

$$\begin{aligned}\Omega_\beta &= (\mathrm{tr}\mathbf{1})\frac{4}{(4\pi)^{\frac{d+1}{2}}}\frac{N_f}{L}\sum_{n=1}^{\infty}(-1)^n\cosh(\mu\beta n)\left[\left(\frac{2\phi}{L\beta n}\right)^{\frac{d+1}{2}}K_{\frac{d+1}{2}}\left(\frac{\beta n}{L}\phi\right)\right.\\ &\left.\quad+\sum_{l=1}^{\infty}\left\{\left(\frac{2(2\pi l+\phi)}{L\beta n}\right)^{\frac{d+1}{2}}K_{\frac{d+1}{2}}\left(\beta n\frac{(2\pi l+\phi)}{L}\right)+[\phi\to-\phi]\right\}\right].\end{aligned} \tag{28}$$

We use an integral representation of the modified Bessel function [20]

$$K_\nu(z) = \frac{\sqrt{\pi}(z/2)^\nu}{\Gamma(\nu+1/2)}\int_1^\infty e^{-zx}(x^2-1)^{\nu-\frac{1}{2}}dx. \tag{29}$$

Then one can perform the summation over n to obtain

$$\begin{aligned}\Omega_\beta &= -(\mathrm{tr}\mathbf{1})\frac{N_f}{L(4\pi)^{\frac{d}{2}}\Gamma(\frac{d+2}{2})}\int_1^\infty dx(x^2-1)^{\frac{d}{2}}\left[\left(\frac{\phi}{L}\right)^{d+1}\left(\frac{1}{\exp[\beta(\frac{\phi x}{L}-\mu)]+1}+(\mu\to-\mu)\right)\right.\\ &\left.\quad+\sum_{l=1}^{\infty}\left\{\left(\frac{2\pi l+\phi}{L}\right)^{d+1}\left(\frac{1}{\exp\left[\beta\left(\frac{(2\pi l+\phi)x}{L}-\mu\right)\right]+1}+(\mu\to-\mu)\right)+[\phi\to-\phi]\right\}\right].\end{aligned} \tag{30}$$

Further, in order to describe a system, which consists of the degenerate fermion gas, *i.e.*, in the situation of $\mu\neq 0$ and $T\to 0$, one can use the fact that, in the zero-temperature limit $T\to 0$ ($\beta\to\infty$),

$$\frac{1}{e^{\beta x}+1}\xrightarrow{\beta\to\infty}\theta(-x), \tag{31}$$

where $\theta(x)$ is the step function. The expression (30) reduces to the following form by means of (31) in the zero-temperature limit

$$\begin{aligned}\Omega_\beta \;=\; & -(\mathrm{tr}\mathbf{1})\frac{N_f}{L(4\pi)^{\frac{d}{2}}\Gamma(\frac{d+2}{2})}\left[\left(\frac{\phi}{L}\right)^{d+1}\int_1^{\mu/\omega_1}(x^2-1)^{\frac{d}{2}}dx\right.\\ & \left.+\sum_{l=1}^{l_m}\left(\frac{2\pi l+\phi}{L}\right)^{d+1}\int_1^{\mu/\omega_2}(x^2-1)^{\frac{d}{2}}dx+\sum_{l=1}^{l_n}\left(\frac{2\pi l-\phi}{L}\right)^{d+1}\int_1^{\mu/\omega_3}(x^2-1)^{\frac{d}{2}}dx\right],\end{aligned} \tag{32}$$

where $\omega_1=\frac{\phi}{L}$, $\omega_2=\frac{2\pi l+\phi}{L}$ and $\omega_3=\frac{2\pi l-\phi}{L}$, and l_m, l_n are the largest integer satisfying $\omega_{2,3}<\mu$, respectively and if $\omega_{2,3}>\mu$, then $\Omega_\beta=0$.

It is interesting to study the case that μ is less than, or at most nearly equals to $2\pi/L$. We only, therefore, consider $0<\mu L<\pi$ case.

In $M_3\times S^1$ space-time, one can express the analytical form:

$$\Omega_\beta=-\frac{N_f}{3}\frac{1}{\pi L^4}(\phi-\mu L)^2(2\phi+\mu L), \tag{33}$$

for $0\le\phi\le\mu L$,

$$\Omega_\beta=0, \tag{34}$$

for $\mu L<\phi\le 2\pi-\mu L$,

$$\Omega_\beta=-\frac{N_f}{3}\frac{1}{\pi L^4}\left(2\pi-\phi-\mu L\right)^2\left(4\pi-2\phi+\mu L\right), \tag{35}$$

for $2\pi-\mu L<\phi<2\pi$.

Further, in $M_4\times S^1$ space-time, one can also represent the analytical form:

$$\Omega_\beta=-\frac{N_f}{12\pi^2L^5}\left[\mu L(2\mu^2L^2-5\phi^2)\sqrt{\mu^2L^2-\phi^2}+3\phi^4\ln\left(\frac{\mu L}{\phi}+\sqrt{\frac{\mu^2L^2}{\phi^2}-1}\right)\right], \tag{36}$$

for $0\le\phi\le\mu L$,

$$\Omega_\beta=0, \tag{37}$$

for $\mu L<\phi\le 2\pi-\mu L$,

$$\begin{aligned}\Omega_\beta \;=\; -\;&\frac{N_f}{12\pi^2L^5}\left[\mu L\{2\mu^2L^2-5(2\pi-\phi)^2\}\sqrt{\mu^2L^2-(2\pi-\phi)^2}\right.\\ &\left.+3\phi^4\ln\left(\frac{\mu L}{2\pi-\phi}+\sqrt{\frac{\mu^2L^2}{(2\pi-\phi)^2}-1}\right)\right],\end{aligned} \tag{38}$$

for $2\pi-\mu L<\phi<2\pi$.

III. CONSTRUCTION OF A VACUUM GAUGE BALL

With the expressions of the thermodynamic potential in the presence of a strongly degenerate fermion gas in $M_3 \times S^1$ and $M_4 \times S^1$ space-times in the previous section, we will show that if the configuration of ϕ is a domain wall configuration, one can construct a vacuum gauge ball, observing that the thermodynamic potential has a barrier between their vacua, and the two vacua are almost degenerate with the energy density difference, which is written as a function of μL, in the case of $0 \leq \phi \leq \mu L$.

A. $M_3 \times S^1$ space-time

We rewrite the thermodynamic potential, from (23) and (33), in the case of $0 \leq \phi \leq \pi$,

$$\Omega = \frac{N_f}{6\pi^2 L^4}\left[2\pi^2(\phi-\pi)^2 - (\phi-\pi)^4\right] - \frac{N_f}{3\pi L^4}(\phi-\mu L)^2(2\phi+\mu L)\theta(\mu L-\phi). \qquad (39)$$

In Fig. 1, we show the thermodynamic potential for various values of μL. At $\phi = 0, \pi$, the values of the thermodynamic potential are given by

$$\Omega = \frac{N_f \pi^2}{6L^4} - \frac{N_f(\mu L)^3}{3\pi L^4}, \qquad \text{at} \quad \phi = 0, \qquad (40)$$

$$\Omega = 0, \qquad \text{at} \quad \phi = \pi, \qquad (41)$$

respectively. There are three cases, which lead to the distinct physical vacua for values of μL. For $0 \leq \mu L < 2^{-3/2}\pi$ case, the thermodynamic potential has a minimum at $\phi = \pi$ only. For $\mu L = 2^{-3/2}\pi$ case, the values of the thermodynamic potential at $\phi = 0$ and $\phi = \pi$ are degenerate. For $\mu L > 2^{-3/2}\pi$ case, the thermodynamic potential has a minimum at $\phi = 0$. If one considers the case of $\mu L \gtrsim 2^{-3/2}\pi$, the two vacua (at $\phi = 0$ and $\phi = \pi$) become almost degenerate. Within the viewpoint of symmetry breaking [3], the fermions, which are massless at the classical level, acquire dynamical masses through quantum corrections for $0 \leq \mu L < 2^{-3/2}\pi$ case, because the physical vacuum has a minimum at the non-trivial configuration $\phi = \pi$. Moreover, for $\mu L > 2^{-3/2}\pi$ case, the dynamical masses of the fermions disappear because the physical vacuum is at the trivial configuration of $\phi = 0$. In general, for non-abelian gauge theory, the different vacua correspond to breaking and restoration of gauge symmetry [6]. Thus, for degenerate fermions, the finite density effects are crucial to investigate whether symmetry is breaking or not.

Now, in order to construct a vacuum gauge ball, we restrict our attention to the case of the almost degenerate vacua ($\mu L \gtrsim 2^{-3/2}\pi$). We should assume that the space-time has a spherical symmetry and further the vacuum gauge ball does not have the $U(1)$ charge, otherwise, the Coulomb force between fermions destroys the strong degenerate state. The situation we assume is whether the gauge coupling constant is very small, or the charge of each fermion, which composes the vacuum gauge ball, compensates with each other. Thus, ϕ is a function of the radial coordinate r only and the Maxwell term in the Lagrangian (1) will be reduced to

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\frac{1}{g^2L^2}(\phi'(r))^2, \tag{42}$$

where $'$ denotes the derivative with respect to r.

For the gauge field, the equation of motion takes the form

$$\frac{1}{r}(r\phi')' - g^2L^2\frac{\partial\Omega}{\partial\phi} = 0, \tag{43}$$

where we have assumed that L is fixed to a constant value.

From (39), thermodynamical quantities at $\beta \to \infty$ are as follows

$$n(r) = -\left.\frac{\partial\Omega}{\partial\mu}\right|_{V_2,L} = \frac{N_f}{\pi L^3}(\mu^2L^2 - \phi^2)\theta(\mu L - \phi), \tag{44}$$

$$\rho(r) = \Omega + \mu n(r) + \frac{1}{2}\frac{1}{g^2L^2}(\phi')^2, \tag{45}$$

$$= \frac{N_f}{3\pi L^4}\left[2(\mu^3L^3 - \phi^3)\theta(\mu L - \phi) + \pi(\phi-\pi)^2 - \frac{1}{2\pi}(\phi-\pi)^4\right] + \frac{1}{2}\frac{1}{g^2L^2}(\phi')^2, \tag{46}$$

where $\rho(r)$ and $n(r)$ are the energy density and the particle number density, respectively. In (45), the first and second terms in the right hand side are the contributions from the fermions (coupled with the vacuum gauge field), and the third term is that from the vacuum gauge field only to the energy density.

By analyzing the field equation (43) numerically, one can search for the configuration of $\phi(r)$, which is the case of a domain wall configuration, as shown in Fig. 2, for a positive value of μL ($\gtrsim 2^{-3/2}\pi$). In this configuration, we show the profiles of the energy density $\rho(r)$ and the particle number density $n(r)$ of the vacuum gauge ball, in Fig. 3. Then, one can obtain the physical values (the mass M and the particle number N) of the vacuum gauge

ball as follows

$$M = 2\pi L \int_0^\infty \rho(r) r dr, \tag{47}$$

$$N = 2\pi L \int_0^R n(r) r dr, \tag{48}$$

where R is a radius of the vacuum gauge ball, and $n(r) = 0$ and $\rho(r) = 0$ for $r > R$.

For the various positive values of μL ($\gtrsim 2^{-3/2}\pi$), we construct the vacuum gauge balls and plot the relationship between the mass $\tilde{M}$ and the particle number $\tilde{N}$ ($\tilde{M} = g^2 LM$ and $\tilde{N} = g^2 N$) in Fig. 4. As far as we can form them numerically, one sees that they are stable energetically because of satisfying the condition of (mass $\times$ particle number) $>$ (energy), where the mass (of the fermion in the vacuum) is π/L, and the dots representing the ball should be connected to the straight line $\tilde{M} = \pi\tilde{N}$, which is similar to the NTS case [14].

In Fig. 5, we show the relation between the particle number $\tilde{N}$ and the radius R. One sees that the larger vacuum gauge balls are formed as the particle numbers increase.

B. $M_4 \times S^1$ space-time

One can perform the construction of the vacuum gauge ball, using the manners in $M_3 \times S^1$ space-time. We rewrite the thermodynamic potential, from (20) and (36), in the case of $0 \le \phi \le \pi$,

$$\Omega = \frac{6N_f}{\pi^2 L^5} \sum_{l=1}^{\infty} \frac{\cos(l\phi) - (-1)^l}{l^5} - \frac{N_f}{12\pi^2 L^5} \Big[\mu L (2\mu^2 L^2 - 5\phi^2) \sqrt{\mu^2 L^2 - \phi^2} + 3\phi^4 \ln \left(\frac{\mu L}{\phi} + \sqrt{\frac{\mu^2 L^2}{\phi^2} - 1} \right) \Big] \theta(\mu L - \phi). \tag{49}$$

At $\phi = 0$, π, the values of the thermodynamic potential are obtained by

$$\Omega = \frac{6N_f}{\pi^2 L^5} \sum_{l=1}^{\infty} \frac{1 - (-1)^l}{l^5} - \frac{N_f (\mu L)^4}{6\pi^2 L^5}, \qquad \text{at} \quad \phi = 0, \tag{50}$$

$$\Omega = 0, \qquad \text{at} \quad \phi = \pi, \tag{51}$$

respectively. There are three cases, which lead to the different physical vacua for values of μL in such case as $M_3 \times S^1$ space-time. At $\mu L = (\frac{279}{4}\zeta(5))^{1/4} \simeq 2.91624$, the two vacua are degenerate, where we have used the facts of $\sum_{l=1}^{\infty} \frac{1}{l^5} = \zeta(5)$ and $\sum_{l=1}^{\infty} \frac{(-1)^l}{l^5} = -\frac{15}{16}\zeta(5)$.

For the vacuum gauge ball, we should take the same assumptions for the symmetry of the space-time and the global charge of the ball. Then, the equation of motion for the vacuum gauge field with the fixed value of L is given by

$$\frac{1}{r^2}(r^2\phi')' - g^2L^2\frac{\partial\Omega}{\partial\phi} = 0. \tag{52}$$

From (49), thermodynamic quantities are as follows

$$\begin{aligned} n(r) &= \frac{2N_f}{3\pi^2L^4}(\mu^2L^2-\phi^2)^{3/2}\theta(\mu L-\phi), && (53)\\ \rho(r) &= \frac{6N_f}{\pi^2L^5}\sum_{l=1}^{\infty}\frac{\cos(l\phi)-(-1)^l}{l^5} + \frac{N_f}{4\pi^2L^5}\Big[\mu L(2\mu^2L^2-\phi^2)\sqrt{\mu^2L^2-\phi^2} \\ &\quad -\phi^4\ln\left(\frac{\mu L}{\phi}+\sqrt{\frac{\mu^2L^2}{\phi^2}-1}\right)\Big]\theta(\mu L-\phi)+\frac{1}{2}\frac{1}{g^2L^2}(\phi')^2. && (54) \end{aligned}$$

By solving the field equation (52) numerically, one can search for a domain wall configuration for ϕ, for a positive value of μL ($\gtrsim (\frac{279}{4}\zeta(5))^{1/4}$), appeared in like Fig. 2. In this configuration, one can obtain the physical values of the vacuum gauge ball as follows

$$\begin{aligned} M &= 4\pi L\int_0^{\infty}\rho(r)r^2dr, && (55)\\ N &= 4\pi L\int_0^{R}n(r)r^2dr, && (56) \end{aligned}$$

where R is a radius of the vacuum gauge ball, and $n(r)=0$ and $\rho(r)=0$ for $r>R$.

For the various positive values of μL ($\gtrsim (\frac{279}{4}\zeta(5))^{1/4}$), we also construct the vacuum gauge ball and display the relationship between the mass $\tilde{M}$ and the particle number $\tilde{N}$ in Fig. 6, where

$$\tilde{M} = \sqrt{\frac{N_f}{6\pi^2}}\left(\frac{g}{L^{\frac{1}{2}}}\right)g^2M, \qquad \tilde{N} = \sqrt{\frac{N_f}{6\pi^2}}\left(\frac{g}{L^{\frac{1}{2}}}\right)^3N. \tag{57}$$

$\frac{g}{L^{\frac{1}{2}}}$ is reduced to the effective four-dimensional gauge coupling. Unlike the case of $M_3\times S^1$ space-time, we see that the dots showing the ball solutions should be connected to the straight line $\tilde{M}=\pi\tilde{N}$, but for some small masses of the balls, the dots cross the line. This situation is also similar to the NTS case. For some small particle number $\tilde{N}$, the balls seem to be energetically unstable, but the balls will be energetically stable since we assume that N_f and N are macroscopic quantity so that $\tilde{N}$ are also macroscopic quantity. And then the values of $\tilde{M}$ are large for macroscopic quantity of $\tilde{N}$ so that the mass of the balls will be heavy.

Further, we plot the relation between the particle number $\tilde{N}$ and the radius R in Fig. 7. One sees that the radius has a minimum value for the non-zero particle number. The reason we think is that if the particle number is small, the contribution from the surrounding wall is larger than that from the fermions to the total energy of the vacuum gauge ball. The actual radius of the ball is given by

$$\sqrt{\frac{6\pi^2}{N_f}} \left(\frac{L^{\frac{1}{2}}}{g} \right) LR. \tag{58}$$

Even if the large values of R are obtained for macroscopic quantity of $\tilde{N}$, form Fig. 7, the circumference of S^1 is set to inverse of some energy scale, for example, $L \sim (\mathrm{TeV})^{-1}$ so that the actual size of the balls will be small. Thus, this small but heavy ball, which does not have the $U(1)$ charge, will be a candidate for unknown matter such as Q-ball, NTS and F-ball.

IV. DISCUSSION

In this paper, we have considered the abelian gauge theory coupled with the Dirac fermion in $M_3 \times S^1$ and $M_4 \times S^1$ space-times. We have calculated the thermodynamic potential for the vacuum gauge field in the presence of a strongly degenerate fermion gas, which is parametrized as μL.

We have constructed the vacuum gauge ball, observing the three conditions: the configuration of ϕ is a domain wall configuration, the thermodynamic potential has a barrier between their vacua and the two vacua are almost degenerate for some values of μL.

Performing the numerical analysis of the field equation for the vacuum gauge field, we have found the relationship between the mass $\tilde{M}$ and the particle number $\tilde{N}$ for various vacuum gauge balls in Fig. 4 and Fig. 6, and between the radius R and the particle number $\tilde{N}$ in Fig. 5 and Fig. 7. For the behavior between $\tilde{M}$ and $\tilde{N}$, the situation we have obtained is similar to the NTS case. But for the behavior between R and $\tilde{N}$ in $M_4 \times S^1$ space-time, the situation is different from that in the NTS case. The reason we think is that when the particle number is small, the contribution from the surrounding wall becomes larger than that from the fermions to the total energy of the vacuum gauge ball. It is still unclear that the behavior between $\tilde{N}$ and R for some small values of $\tilde{N}$ differs in $M_3 \times S^1$ and $M_4 \times S^1$ space-times.

Although we have not investigated the cases of non-abelian gauge theory and other topology of extra dimensions, the vacuum gauge balls we have constructed may be sufficient to know the fundamental properties of the ball, which is constructed from more complicated situations.

So far, our analysis is confined to the flat background space-time. It would also be interesting to consider the vacuum gauge ball in curved space-time, such as Q-star [21] and Boson star (for a review, see Ref. [22]), and to construct the ball in the little Higgs model (or a model with dimensional deconstruction [23]). Further, in order to investigate the stability of the vacuum gauge balls, the calculation of quantum correction to the their energy is very important. We must continue to make every effort to study these situations.

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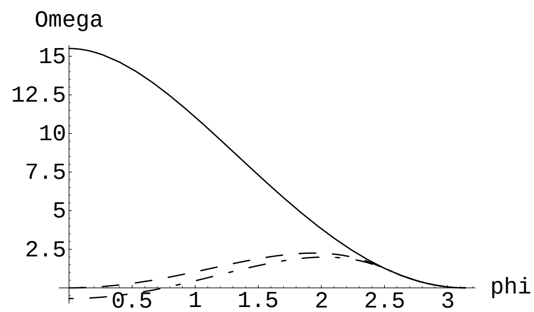

FIG. 1: The thermodynamic potential Ω of (39) is displayed against ϕ for various values of μL in $M_3 \times S^1$ space-time. The solid line corresponds to $\mu L = 0$, the dashed line corresponds to $\mu L = 2^{-3/2}\pi$ (two degenerate vacuum case), and the dot-dashed line corresponds to $\mu L = 2.5$ ($\gtrsim 2^{-3/2}\pi$) (two almost degenerate vacuum case).

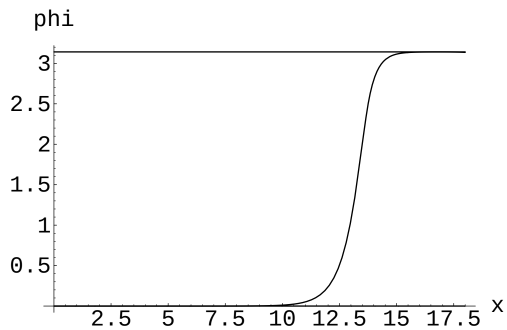

FIG. 2: The configuration of $\phi(x)$, which is a domain wall configuration, is shown against x $(= \sqrt{\frac{N_f}{3\pi}\frac{gr}{L}})$.

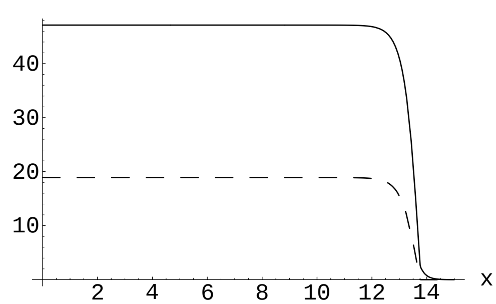

FIG. 3: The profiles of the energy density $\rho(x)$ and the particle number density $n(x)$ are shown against x $(= \sqrt{\frac{N_f}{3\pi}\frac{gr}{L}})$ in the case that the configuration of ϕ is a domain wall configuration. The solid line corresponds to $\rho(x)$ and the dashed line corresponds to $n(x)$.

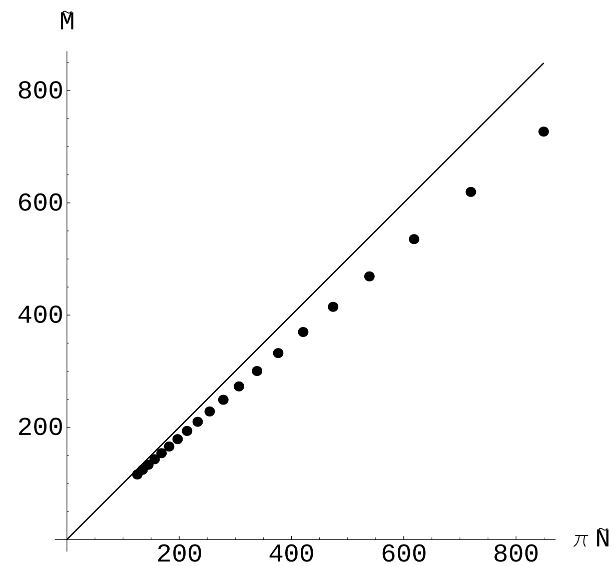

FIG. 4: For the various positive values of μL ($\gtrsim 2^{-3/2}\pi$), the relationship between the mass $\tilde{M}$ and the particle number $\tilde{N}$ of the vacuum gauge ball is shown, where $\tilde{M}$ and $\tilde{N}$ are defined as $\tilde{M} \equiv g^2 LM$ and $\tilde{N} \equiv g^2 N$, respectively in $M_3 \times S^1$ space-time. The straight line corresponds to $\tilde{M} = \pi\tilde{N}$.

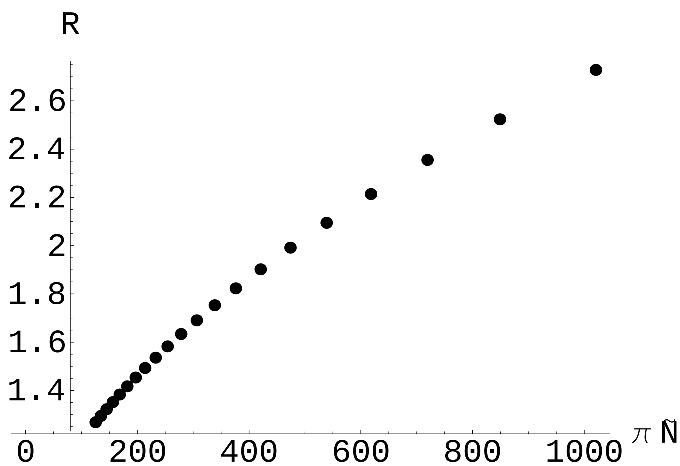

FIG. 5: For the various positive values of μL ($\gtrsim 2^{-3/2}\pi$), the relationship between the radius R and the particle number $\tilde{N}$ for the vacuum gauge ball is shown in $M_3 \times S^1$ space-time.

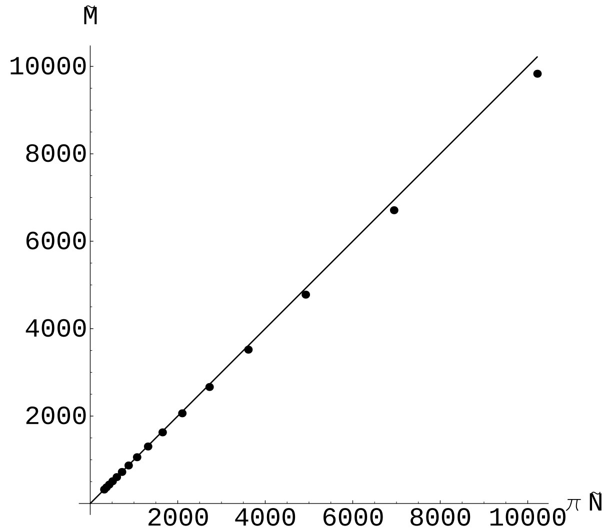

FIG. 6: For the various positive values of μL ($\gtrsim (\frac{279}{4}\zeta(5))^{1/4}$), the relationship between the mass $\tilde{M}$ and the particle number $\tilde{N}$ for the vacuum gauge ball is shown, where $\tilde{M}$ and $\tilde{N}$ are defined as $\tilde{M} \equiv \frac{g^3}{\pi}\sqrt{\frac{N_f}{6L}}M$ and $\tilde{N} \equiv \frac{g^3}{\pi L^{3/2}}\sqrt{\frac{N_f}{6}}N$, respectively in $M_4 \times S^1$ space-time. The straight line corresponds to $\tilde{M} = \pi\tilde{N}$.

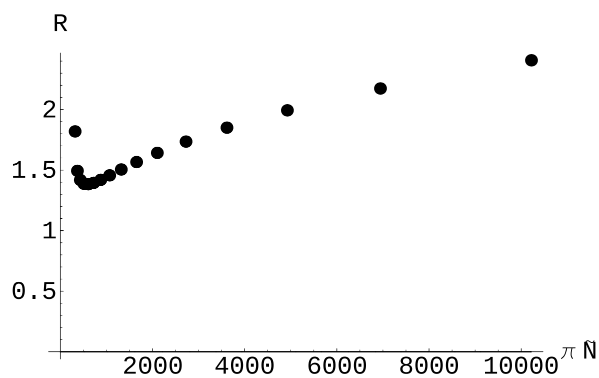

FIG. 7: For the various positive values of μL ($\gtrsim (\frac{279}{4}\zeta(5))^{1/4}$), the relationship between the radius R and the particle number $\tilde{N}$ for the vacuum gauge ball is shown in $M_4 \times S^1$ space-time.